XX. *On the Probabilities of Survivorſhips between Two Perſons of any given Ages, and the Method of determining the Values of Reverſions depending on thoſe Survivorſhips. By Mr.* William Morgan; *communicated by the Rev.* Richard Price, *D. D. F. R. S.*

Read May 8, 1788.

THE hypotheſis of an equal decrement of life, adopted by M. DE MOIVRE, for the purpoſe of facilitating the computations of life annuities, has not only been rendered unneceſſary by the late publication of many excellent tables deduced from real obſervations *, but has likewiſe been found ſo very incorrect in ſome caſes, that probably little or no recourſe will ever be had to it in future. But though the direct application of this hypotheſis may be laid aſide, there is danger of its not being entirely abandoned; and mathematicians may ſtill be led to reaſon from this principle, by deriving their rules from the *expectations* rather than from the *real probabilities* of life. The ingenious Mr. THOMAS SIMPSON has contented himſelf with this inaccurate method in his Select Exerciſes, and he has been followed in it by moſt other writers on the ſubject †. Even in thoſe caſes which involve only two lives, the errors are often

* See Dr. PRICE's Treatiſe on Reverſionary Payments, 4th edit.; and Mr. Baron MASERES on Life Annuities.

† Dr. PRICE's ſolutions of his 15th and 16th queſtions are exceptions to this remark.

confiderable, efpecially if the expectations are derived from the *London*, the *Sweden*, or any other tables, in which the decrements of life are unequal. But when three lives are involved in the queftion, thefe errors are generally enormous; nor is it ever fafe, when the ages of thofe lives differ very much, to have recourfe to rules which are founded upon this principle. The three following problems, though the moft common in the doctrine of furvivorfhips, have never hitherto been folved in a manner ftrictly true. The fecond of them is of particular importance, and I have taken much pains to examine how far Mr. SIMPSON's folution of it may be depended upon. It has, indeed, been folved by M. DE MOIVRE *, and Mr. DODSON †: but the firft of thefe writers has erred moft egregioufly in the folution itfelf, and the other having derived his rule from a wrong hypothefis, has rendered it of no ufe. It is much to be wifhed that the folutions of all cafes in reverfions and furvivorfhips were deduced, like the three following ones, from the real probabilities of life. Moft of thofe which are now in ufe are at beft but approximations, and can never be relied on with any tolerable degree of fatisfaction.

PROBLEM I.

Suppofing the ages of two perfons, A and B, to be given; to determine the probabilities of furvivorfhip between them from any table of obfervations.

SOLUTION.

Let a reprefent the number of perfons living in the table at the age of A the younger of the two lives. Let a', a'', a''',

* See Mr. DE MOIVRE's 17th problem, and Dr. PRICE's remarks upon it in his Treatife on Reverfionary Payments, Effay 3. Vol. I.

† See DODSON's Mathematical Repofitory, Prob. 23. Vol. III.

a'''', &c. repreſent the decrements of life at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of A. Let b repreſent the number of perſons living at the age of B the older of the two lives, and c, d, e, f, &c. the number of perſons living at the end of the 1ſt, 2d, 3d, 4th, &c. years from the age of B. Suppoſing now it were required to determine the probability of B's ſurviving A in the firſt year. It is manifeſt that this event may take place either by A's dying before the end of the year and B's ſurviving that period, or by the extinction of both the lives, reſtrained however to the contingency of B's having died *laſt*. The probability that A dies in the firſt year, and that B ſurvives it, is expreſſed by the fraction $\frac{a'c}{ab}$. The probability that both the lives die in this year is expreſſed by the fraction $\frac{a' \cdot \overline{b-c}}{ab}$; and as it is very nearly an equal chance that A dies *firſt*, this fraction ſhould be reduced *one-half*, and then it will become $= \frac{a' \cdot \overline{b-c}}{2ab}$. Hence the whole probability of B's ſurviving A in the firſt year will be $= \frac{a'c}{ab} + \frac{a' \cdot \overline{b-c}}{2ab} = \frac{a' \cdot \overline{b+c}}{2ab}$. In the ſame manner the probability of B's ſurving A in the 2d, 3d, 4th, &c. years may be found $= \frac{a'' \cdot \overline{c+d}}{2ab} \ldots \frac{a''' \cdot \overline{d+e}}{2ab} \ldots \frac{a'''' \cdot \overline{e+f}}{2ab}$, &c. reſpectively; therefore, the whole probability of B's ſurviving A will be $= \frac{1}{ab} \times \overline{\frac{b+c}{2} a' + \frac{c+d}{2} a'' + \frac{d+e}{2} a''' + \frac{e+f}{2} a''''}$, &c.

Having found by the preceding ſeries the probability of B, the elder life's, ſurviving A the younger; the other expreſſion, which

which denotes the probability of A's ſurviving B, is well known to be the difference between the foregoing ſeries and *unity*.

The ſum of this ſeries might eaſily be determined from tables of the expectations of ſingle and joint lives*. But no ſuch table as the latter having ever been computed, it will by no means be found a laborious undertaking to compute a table of the probabilities of ſurvivorſhip between two perſons of all ages immediately from this ſeries, without having recourſe to the expectations of life. For if the probability of ſurvivorſhip between any two perſons be found, the probability between two perſons one year younger is obtained with little difficulty; and by proceeding in this manner a whole table may be formed in leſs time than would be neceſſary for computing one of the expectations of two joint lives. To exemplify what I have ſaid, I ſhall juſt ſet down a few operations for determining the probability of ſurvivorſhip, according to the Northampton Table of Obſervations †, between two perſons, whoſe common difference of age is 10 years.

* The above ſeries, expreſſing the probability of ſurvivorſhip, may be eaſily found, from the ſolution of the next problem, $= \frac{\beta \cdot \overline{K - AK} - c \cdot \overline{C - AC}}{2b}$; K denoting the expectation of one life one year younger than B; C the expectation of a life one year older than B; AK and AC the expectations of the joint lives of A and K, and of A and C, reſpectively; and β, as in that problem, the number of perſons living in the table of obſervations at the age of K.

† See Dr. PRICE's Treatiſe on Reverſionary Payments, Tab. 6. Vol. II.

Age

Age of B.	Age of A.	Probability of B's ſurviving A.	Probability of A's ſurviving B.
96	86	$\frac{1}{1\times 145}\times\frac{\overline{0+1}}{2}\times 34 = .1173$ . .	$1 - .1173 = .8887$
95	85	$\frac{1}{4\times 186}\times\frac{\overline{4+1}}{2}\times 41 + 17 = .1606$	$1 - .1606 = .8394$
94	84	$\frac{1}{9\times 234}\times\frac{\overline{9\times 4}}{2}\times 48 + 119.5 = .2049$	$1 - .2049 = .7951$
93	83	$\frac{1}{16\times 289}\times\frac{\overline{16+9}}{2}\times 55 + 431.5 = .2420$	$1 - .2420 = .7580$
92	82	$\frac{1}{24\times 346}\times\frac{\overline{24+16}}{2}\times 57 + 1119 = .2720$	$1 - .2720 = .7280$
91	81	$\frac{1}{34\times 406}\times\frac{\overline{34+24}}{2}\times 60 + 2259 = .2897$	$1 - .2897 = .7103$
90	80	$\frac{1}{46\times 469}\times\frac{\overline{46+34}}{2}\times 63 + 3999 = .3022$	$1 - .3022 = .6978$

It may eaſily be ſeen from theſe ſpecimens in what manner the probabilities of ſurvivorſhip between two younger lives are deduced from the probabilities between two older lives, provided their common difference of age be the ſame; for the numbers 17 . . 119.5 . . . 431.5, &c. in the 2d, 3d, 4th, &c. ſeries are the ſums of the ſeries next preceding. Thus 17 is $= 34 \times \frac{1}{2}$ . . . 119.5 is $= 41 \times \frac{5}{2} + 17$ . . . . 431.5 is $= 48 \times \frac{13}{2} + 119.5$, &c. It may be neceſſary to obſerve further, that if the ages of the two perſons be equal, the probability of ſurvivorſhip between them being likewiſe equal, is expreſſed by the fraction $\frac{1}{2}$; and that this affords an inſtance of the accuracy of the foregoing inveſtigation; for the ſeries expreſſing the probability in this caſe is the ſame with this fraction, the chance of ſurvivorſhip becoming then (ſince $a = b$; $a' = b - c$; $a'' = c - d$, &c.; and $\overline{b+c} \times a' = \overline{b+c} \times \overline{b-c} = b^2 - c^2$, &c.) $= \frac{bb-cc}{2bb} + \frac{cc-dd}{2bb}$, &c. $= \frac{1}{2}$.

Mr.

Mr. SIMPSON, in his Treatiſe on Annuities and Reverſions * has given a curve whoſe area determines the probability of ſurvivorſhip between two perſons according to any table of obſervations. If one of the lives be not very young ſo that the equidiſtant ordinates may not be too few, this area is ſufficiently correct. But if the eldeſt of the two lives is under 20 years of age, it becomes neceſſary to aſſume ſo many equidiſtant ordinates to render the ſolution accurate, when the decrements of life are unequal, that the operation is rendered much too laborious for uſe; nor do I know that it can be neceſſary to have recourſe to this area in any caſe, eſpecially as the true probabilities of ſurvivorſhip are ſo eaſily computed from the preceding ſeries.

The following table has been formed in the manner deſcribed above; and as no ſuch table has ever been attempted before, I have been the more deſirous to render it complete, by computing the probabilities of ſurvivorſhip between two perſons of all ages, whoſe common difference is not leſs than ten years.

Inſtead alſo of ſuppoſing certainty to be denoted by unity, I have aſſumed 100 for this purpoſe; ſo that the ſums in the adjoining columns expreſs the number of chances in 100 which are in favour of B or A's ſurviving the other.

* Lemma 2, p. 100.

TABLE,

Table, shewing the probabilities of survivorship between two persons of all ages, whose common difference of age is not less than ten years, computed from the Northampton Table of Observations in Dr. Price's Treatise on Reversionary Payments.

Ten years difference.			Twenty years difference.			Thirty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
11 — 1	58.59	41.41	21 — 1	52.46	47.54	31 — 1	48.23	51.77
12 — 2	51.36	48.64	22 — 2	44.33	55.67	32 — 2	39.36	60.64
13 — 3	48.23	51.77	23 — 3	40.88	59.12	33 — 3	35.57	64.43
14 — 4	45.98	54.02	24 — 4	39.00	61.00	34 — 4	32.86	67.14
15 — 5	44.70	55.30	25 — 5	37.69	62.31	35 — 5	31.35	68.65
16 — 6	43.43	56.57	26 — 6	36.40	63.60	36 — 6	29.85	70.15
17 — 7	42.53	57.47	27 — 7	35.48	64.52	37 — 7	28.83	71.17
18 — 8	41.91	58.09	28 — 8	34.84	65.16	38 — 8	27.98	72.02
19 — 9	41.60	58.40	29 — 9	34.52	65.48	39 — 9	27.53	72.47
20 — 10	41.53	58.47	30 — 10	34.41	65.59	40 — 10	27.33	72.67
21 — 11	41.58	58.42	31 — 11	34.40	65.60	41 — 11	27.24	72.76
22 — 12	41.68	58.32	32 — 12	34.42	65.58	42 — 12	27.18	72.82
23 — 13	41.78	58.22	33 — 13	34.44	65.56	43 — 13	27.14	72.86
24 — 14	41.90	58.10	34 — 14	34.47	65.53	44 — 14	27.11	72.89
25 — 15	42.02	57.98	35 — 15	34.51	65.49	45 — 15	27.08	72.92
26 — 16	42.16	57.84	36 — 16	34.56	65.44	46 — 16	27.06	72.9[illegible]
27 — 17	42.26	57.74	37 — 17	34.58	65.42	47 — 17	27.01	72.99
28 — 18	42.32	57.68	38 — 18	34.54	65.46	48 — 18	26.90	73.10
29 — 19	42.34	57.66	39 — 19	34.44	65.56	49 — 19	26.72	73.28
30 — 20	42.31	57.69	40 — 20	34.29	65.71	50 — 20	26.50	73.50
31 — 21	42.22	57.78	41 — 21	34.08	65.92	51 — 21	26.21	73.79
32 — 22	42.09	57.91	42 — 22	33.84	66.16	52 — 22	25.89	74.11
33 — 23	41.97	58.03	43 — 23	33.59	66.41	53 — 23	25.56	74.44
34 — 24	41.83	58.17	44 — 24	33.35	66.65	54 — 24	25.22	74.78
35 — 25	41.70	58.30	45 — 25	33.09	66.91	55 — 25	24.87	75.13
36 — 26	41.56	58.44	46 — 26	32.83	67.17	56 — 26	24.52	75.48
37 — 27	41.41	58.59	47 — 27	32.56	67.44	57 — 27	24.16	75.84
38 — 28	41.26	58.74	48 — 28	32.28	67.72	58 — 28	23.79	76.21
39 — 29	41.10	58.90	49 — 29	31.99	68.01	59 — 29	23.41	76.59
40 — 30	40.94	59.06	50 — 30	31.70	68.30	60 — 30	23.02	76.98
41 — 31	40.78	59.22	51 — 31	31.43	68.57	61 — 31	22.63	77.37
42 — 32	40.63	59.37	52 — 32	31.16	68.84	62 — 32	22.23	77.77
43 — 33	40.49	59.51	53 — 33	30.88	69.12	63 — 33	21.81	78.19

Ten years difference.			Twenty years difference.			Thirty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
44—34	40.34	59.66	54—34	30.60	69.40	64—34	21.38	78.62
45—35	40.19	59.81	55—35	30.32	69.68	65—35	20.93	79.07
46—36	40.04	59.96	56—36	30.03	69.97	66—36	20.47	79.53
47—37	39.88	60.12	57—37	29.73	70.27	67—37	20.01	79.99
48—38	39.71	60.29	58—38	29.43	70.57	68—38	19.54	80.46
49—39	39.54	60.46	59—39	29.13	70.87	69—39	19.06	80.94
50—40	39.38	60.62	60—40	28.82	71.18	70—40	18.59	81.41
51—41	39.23	60.77	61—41	28.49	71.51	71—41	18.10	81.90
52—42	39.07	60.93	62—42	28.14	71.86	72—42	17.58	82.42
53—43	38.90	61.10	63—43	27.74	72.26	73—43	17.05	82.95
54—44	38.81	61.19	64—44	27.33	72.67	74—44	16.53	83.47
55—45	38.53	61.47	65—45	26.90	73.10	75—45	16.04	83.96
56—46	38.35	61.65	66—46	26.46	73.54	76—46	15.60	84.40
57—47	38.16	61.84	67—47	26.01	73.99	77—47	15.15	84.85
58—48	37.97	62.03	68—48	25.55	74.45	78—48	14.68	85.32
59—49	37.77	62.23	69—49	25.09	74.91	79—49	14.16	85.84
60—50	37.56	62.44	70—50	24.61	75.39	80—50	13.61	86.39
61—51	37.30	62.70	71—51	24.06	75.94	81—51	13.04	86.96
62—52	37.00	63.00	72—52	23.49	76.51	82—52	12.46	87.54
63—53	36.68	63.32	73—53	22.91	77.09	83—53	11.94	88.06
64—54	36.34	63.66	74—54	22.35	77.65	84—54	11.60	88.40
65—55	35.97	64.03	75—55	21.81	78.19	85—55	11.30	88.70
66—56	35.58	64.42	76—56	21.31	78.69	86—56	11.04	88.96
67—57	35.17	64.83	77—57	20.80	79.20	87—57	10.80	89.20
68—58	34.74	65.26	78—58	20.24	79.76	88—58	10.63	89.37
69—59	34.30	65.70	79—59	19.59	80.41	89—59	10.25	89.75
70—60	33.85	66.15	80—60	18.90	81.10	90—60	9.65	90.35
71—61	33.38	66.62	81—61	18.23	81.77	91—61	8.68	91.32
72—62	32.90	67.10	82—62	17.57	82.43	92—62	7.54	92.46
73—63	32.46	67.54	83—63	17.03	82.97	93—63	6.18	93.82
74—64	32.04	67.96	84—64	16.73	83.27	94—64	4.93	95.07
75—65	31.70	68.30	85—65	16.55	83.45	95—65	3.68	96.32
76—66	31.45	68.55	86—66	16.29	83.71	96—66	2.58	97.42
77—67	31.20	68.80	87—67	16.38	83.62			
78—68	30.90	69.10	88—68	16.44	83.56			
79—69	30.47	69.53	89—69	16.23	83.77			
80—70	30.00	70.00	90—70	15.67	84.33			
81—71	29.58	70.42	91—71	14.50	85.50			
82—72	29.19	70.81	92—72	13.05	86.95			
83—73	28.90	71.10	93—73	11.08	88.92			
84—74	29.02	70.98	94—74	9.24	90.76			
85—75	29.17	70.83	95—75	7.17	92.83			
86—76	29.25	70.75	96—76	5.12	94.88			

Ten

Ten years difference.

Ages.	Probabilities.	
87—77	29.42	70.58
88—78	29 94	70.06
89—79	30.29	69.71
90—80	30.22	69.78
91—81	28.97	71.03
92—82	27.20	72.80
93—83	24.20	75.80
94—84	20.49	79.51
95—85	16.06	83.94
96—86	11.73	88.27

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
41— 1	43.57	56.43	51— 1	39.16	60.84	61— 1	34.94	65.06
42— 2	33.98	66.02	52— 2	28.87	71.13	62— 2	24.13	75.87
43— 3	29.85	70.15	53— 3	24.46	75.54	63— 3	19.52	80.48
44— 4	26.88	73.12	54— 4	21.28	78.72	64— 4	16.19	83.81
45— 5	25.20	74.80	55— 5	19.48	80.52	65— 5	14.28	85.72
46— 6	23.53	76.47	56— 6	17.68	82.32	66— 6	12.37	87.63
47— 7	22.30	77.70	57— 7	16.35	83.65	67— 7	10.92	89.08
48— 8	21.40	78.60	58— 8	15.35	84.65	68— 8	9.82	90.18
49— 9	20.86	79.14	59— 9	14.74	85.26	69— 9	9.12	90.88
50—10	20.59	79.41	60—10	14.42	85.58	70—10	8.73	91.27
51—11	20.45	79.55	61—11	14.22	85.78	71—11	8.46	91.54
52—12	20.35	79.65	62—12	14 06	85.94	72—12	8.25	91.75
53—13	20.26	79.74	63—13	13.90	86.10	73—13	8.05	91.95
54—14	20.18	79.82	64—14	13.76	86.24	74—14	7.88	92.12
55—15	20.11	79.89	65—15	13.62	86.38	75—15	7.75	92.25
56—16	20.05	79.95	66—16	13.50	86.50	76—16	7.68	92.32
57—17	19 96	80.04	67—17	13.35	86 65	77—17	7.59	92.41
58—18	19.81	80.19	68—18	13.14	86.86	78—18	7.43	92.57
59—19	19.59	80.41	69—19	12.86	87.14	79—19	7.19	92.81
60—20	19.31	80.69	70—20	12.53	87.47	80—20	6.90	93.10
61—21	18.96	81 04	71—21	12.11	87.89	81—21	6.55	93.45
62—22	18.55	81.45	72—22	11.65	88.35	82—22	6.16	93.84
63—23	18.12	81.88	73—23	11.20	88.80	83—23	5.81	94.19
64—24	17.68	82.32	74—24	10.75	89.25	84—24	5.56	94.44

Forty years difference.			Fifty years difference.			Sixty years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
65—25	17.22	82.78	75—25	10.32	89.68	85—25	5.32	94.68
66—26	16.76	83.24	76—26	9.91	90.09	86—26	5.11	94.89
67—27	16.29	83.71	77—27	9.50	90.50	87—27	4.90	95.10
68—28	15.81	84.19	78—28	9.07	90.93	88—28	4.73	95.27
69—29	15.32	84.68	79—29	8.61	91.39	89—29	4.48	95.52
70—30	14.83	85.17	80—30	8.15	91.85	90—30	4.13	95.87
71—31	14.34	85.66	81—31	7.70	92.30	91—31	3.63	96.37
72—32	13.84	86.16	82—32	7.27	92.73	92—32	3.10	96.90
73—33	13.35	86.65	83—33	6.88	93.12	93—33	2.48	97.52
74—34	12.87	87.13	84—34	6.60	93.40	94—34	1.94	98.06
75—35	12.42	87.58	85—35	6.35	93.65	95—35	1.40	98.60
76—36	11.99	88.01	86—36	6.12	93.88	96—36	0.95	99.05
77—37	11.56	88.44	87—37	5.91	94.09			
78—38	11.12	88.88	88—38	5.74	94.26			
79—39	10.64	89.36	89—39	5.48	94.52			
80—40	10.21	89.79	90—40	5.11	94.89			
81—41	9.68	90.32	91—41	4.55	95.45			
82—42	9.19	90.81	92—42	3.92	96.08			
83—43	8.76	91.24	93—43	3.15	96.85			
84—44	8.46	91.54	94—44	2.47	97.53			
85—45	8.18	91.82	95—45	1.80	98.20			
86—46	7.95	92.05	96—46	1.23	98.77			
87—47	7.74	92.26						
88—48	7.59	92.41						
89—49	7.32	92.68						
90—50	6.90	93.10						
91—51	6.17	93.83						
92—52	5.33	94.67						
93—53	4.32	95.68						
94—54	3.42	96.58						
95—55	2.51	97.49						
96—56	1.73	98.27						

Seventy

Seventy years difference.			Eighty years difference.			Ninety years difference.		
Ages.	Probabilities.		Ages.	Probabilities.		Ages.	Probabilities.	
71 – 1	30.49	69.51	81 – 1	24.95	75.05	91 – 1	18.93	81 07
72 – 2	19.45	80.55	82 – 2	14.37	85.63	92 – 2	9.18	90.82
73 – 3	14.87	85.13	83 – 3	10.34	89.66	93 – 3	5.53	94.46
74 – 4	11.59	88.41	84 – 4	7.63	92.37	94 – 4	3.12	96.88
75 – 5	9.80	90 20	85 – 5	6.34	93.66	95 – 5	2.12	97.88
76 – 6	7.98	92.02	86 – 6	4.92	95.08	96 – 6	1.15	98.85
77 – 7	6.60	93.40	87 – 7	3.85	96.15			
78 – 8	5.53	94.47	88 – 8	3.04	96.96			
79 – 9	4.84	95.16	89 – 9	2.50	97.50			
80 – 10	4.44	95.56	90 – 10	2.16	97.84			
81 – 11	4.18	95.82	91 – 11	1.86	98.14			
82 – 12	3.97	96.03	92 – 12	1.57	98.43			
83 – 13	3.80	96.20	93 – 13	1.25	98.75			
84 – 14	3.72	96.28	94 – 14	0.97	99.03			
85 – 15	3.68	96.32	95 – 15	0.70	99.30			
86 – 16	3.70	96.30	96 – 16	0.49	99.51			
87 – 17	3.73	96.27						
88 – 18	3.75	96.25						
89 – 19	3.67	96.33						
90 – 20	3.45	96.55						
91 – 21	3.09	96.91						
92 – 22	2.63	97.37						
93 – 23	2.10	97.90						
94 – 24	1.64	98.36						
95 – 25	1.18	98.82						
96 – 26	0.80	99.20						

PROBLEM II.

Suppofing the ages of A and B to be given; to determine, from any table of obfervations, the prefent value of the fum S payable on the contingency of one life's furviving the other.

SOLUTION.

Let r denote £ 1. increased by its interest for a year, and let all the other symbols be the same as in the preceding problem. Let the life of B also be supposed to be the older of the two lives; and then it will follow, by reasoning as in the solution of that problem, that the present value of S to be received on death of A, should that happen in the life-time of B, will be expressed by the series $S \times \overline{\frac{\overline{b+c} \cdot a'}{2abr} + \frac{\overline{c+d} \cdot a''}{2abr^2} + \frac{\overline{d+e} \cdot a'''}{2abr^3} + \frac{\overline{e+f} \cdot a''''}{2abr^4}}$ &c. This series may be resolved into the two following; $\frac{S}{2} \times \overline{\frac{ca'}{abr} + \frac{da''}{abr^2} + \frac{ea'''}{abr^3} + \frac{fa''''}{abr^4}}$ &c. $+ \frac{S}{2} + \overline{\frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^2} + \frac{fa''''}{abr^4}}$ &c. The first of these two series may be again resolved into $\frac{S}{2} \times \overline{\frac{c}{br} - \frac{ca - ca'}{abr} + \frac{d}{br^2} - \frac{da - da' - da''}{abr^2} + \frac{e}{br^3} - \frac{ea - ea' - ea'' - ea'''}{abr^3}}$ &c. $\left(-\frac{S}{2} \times \overline{\frac{da'}{abr^2} + \frac{ea' + ea''}{abr^3}}\right.$ &c. $=\left.\right) - S \times \frac{c}{2br} \times \overline{\frac{d}{cr} - \frac{da - da'}{acr} + \frac{e}{cr^2} - \frac{ea - ea' - ea''}{acr^2}}$ &c. Let B denote the value of an annuity on the life of B, C the value of an annuity on a life one year older than B, AB and AC the values of annuities on the joint lives of A and B and of A and C, and these series will be $= \frac{S \times \overline{B - AB}}{2} - \frac{S \times c \times \overline{C - AC}}{2br}$. Again, the second series above mentioned, or $\frac{S}{2} \times \overline{\frac{ba'}{abr} + \frac{ca''}{abr^2} + \frac{da'''}{abr^3}}$ &c., by pursuing the same steps may be found $= \frac{\beta \times S}{2b} \times \overline{K - AK} - \frac{S \cdot \overline{B - AB}}{2r}$ where β denotes the number of persons living at the age of a person one year younger than B, K the value of an annuity on that life, and

and AK the value of an annuity on the joint lives of A and K. The whole value of the ſurvivorſhip is therefore =

$$S \times \overline{\frac{\overline{r-1} \,.\, \overline{B-AB}}{2r} + \frac{\beta \,.\, \overline{K-AK}}{2b} - \frac{c \,.\, \overline{A-AC}}{2br}}. \quad \text{Q. E. D.}$$

Having now the value of the ſum S payable on the contingency of B's ſurviving A, the value of the ſame ſum, payable on the contingency of A's ſurviving B, is eaſily obtained by the well known method of ſubtracting the value found above from the whole value of the reverſion after the extinction of the joint lives of A and B. It is evident, that the exactneſs of the above rule muſt depend on the accuracy with which the values of the ſingle and joint lives are computed. Being poſſeſſed of ſuch tables for all ages, even with reſpect to the joint lives, I have computed the following values, in order that it may be ſeen how far Mr. SIMPSON's approximation * (the only rule now in uſe) may be depended on.

* It muſt be remembered, that the correction explained by Dr. PRICE, in Vol. I. p. 39, &c. of his Treatiſe on Reverſionary Payments, muſt be applied to Mr. SIMPSON's rule; that is, when the reverſion is a *ſum* and not an eſtate, the value found by the rule muſt be divided by £. 1, increaſed by its intereſt for a year.

Age

Age of B.	Age of A.	* Value of £. 100 payable on the death of A if B survives him.		Age of B.	Age of A.	Value of £. 100 payable on the death of A if B survives him.		Age of B.	Age of A.	Value of £. 100 payable on the death of A if B survives him.	
		True value.	SIMPSON's value.			True value.	SIMPSON's value.			True value.	SIMPSON's value.
10	2	32.67	26.05	40	10	17.10	19.07	60	60	38.88	38.92
10	10	24.74	24.75	40	40	32.86	32.87	70	2	18.36	9.81
20	2	29.99	24.73	50	2	23.36	17.06	70	10	7.07	9.15
20	10	22.11	23.50	50	10	14.01	16.21	70	40	15.35	15.78
20	20	27.95	27.96	50	20	18.65	19.29	70	70	42.34	42.33
30	2	28.79	22.60	50	50	35.80	35.85	80	2	14.46	5.71
30	10	19.84	21.47	60	2	21.52	13.61	80	10	3.93	5.43
30	30	30.22	30.21	60	10	10.65	12.93	80	50	12.05	12.00
40	2	26.65	20.07	60	30	17.51	18.19	80	80	45.47	45.45

From this table it appears, that Mr. SIMPSON's approximation in the middle ſtages of life is ſufficiently accurate; but that it is exceedingly defective when the life of A is very young. It ſhould alſo be remembered, that theſe values have been computed at a low rate of intereſt, and from the *Northampton* Table of Obſervations, in which the decrements of life come nearer to M. DE MOIVRE's hypotheſis than in any other table. But if the computations be made at a higher rate of intereſt even from this table, the approximation does not always agree ſo well, as will appear from the following ſpecimens calculated at 5 *per cent.*

* Theſe values have been computed at 3 *per cent.* and from the Northampton Table of Obſervations.

Age

Age of B	Age of A	Value of £. 100 payable on the death of A if B ſurvives him, by the Northampton table.		Value of £. 100 payable on the death of A if B ſurvives him, according to the Sweden Table of Obſervations, and at 4 *per cent.*							
		True value.	SIMPSON's value.	Age of B.	Age of A.	True value	SIMPSON's value.	Age of B.	Age of A.	True value.	SIMPSON's value.
20	2	25.09	18.46	20	2	21.47	16.80	60	36	12.29	16.81
20	10	15.49	17.54	20	14	15.42	17.82	60	42	16.11	19.58
40	2	23.57	15.97	20	20	20.01	19.84	60	60	36.96	36.34
60	2	19.83	11.75	40	4	23.53	14.22	76	40	9.21	9.81
60	40	18.73	19.61*	40	16	13.71	16.23	76	52	12.58	14.00
				40	28	17.60	20.44	76	64	23.81	22.81
				40	40	27.62	27.00	76	76	42.90	43.29
				60	24	9.39	13.01				

In order further to compare Mr. SIMPSON's approximation with the true value, I have inſerted in the foregoing table a few computations deduced from the *Sweden* Table of Obſervations, in which the decrements of life are unequal. From theſe inſtances the approximation appears to be more defective in proportion as the probabilities of life differ from the hypotheſis. It may be proper, however, to obſerve, that this table has been computed from the values of two joint lives given by Dr. PRICE in his Treatiſe on Reverſionary Payments, which values cannot be found with perfect correctneſs at every age, becauſe given in that Treatiſe (Vol. II. p. 57, &c. and p. 144, &c.) only for ages that are either equal, or whoſe difference is five years or ſome multiple of five as in the *Northampton* Table, or ſix years or ſome multiple of ſix as in the *Sweden* Table. This circumſtance may perhaps be ſuppoſed to affect the values of the ſurvivorſhips computed from them, when the

* Theſe values at 3 *per cent.* are 21.92 and 22.78.

joint lives differ from each other in a greater or lefs degree than they have been given in thofe Tables. But by comparing the values of the reverfions when the ages are equal (in which cafe Mr. SIMPSON gives the true rule) it will be feen, that the values of the joint lives are fufficiently correct for the purpofe; and in order to put this matter out of all doubt, I have computed the values by the foregoing rule from the *exact* values of the joint lives by the *Northampton* Table, and alfo from thofe values which have been deduced from Dr. PRICE's Tables, and I have found them to agree exceedingly well *.

It may be perceived, that in all the cafes of *equal* lives I have computed the values of the furvivorfhips by the preceding rule. This has only been done to prove its accuracy by comparing it with Mr. SIMPSON's rule, which is in this particular cafe, as I have already obferved, perfectly right; for when the ages are equal, the chance of furvivorfhip muft alfo be equal, and therefore *half* the value of the reverfion after the extinction of the joint lives will be the true value of the given fum payable on the death of either A or B, fubject to the contingency of his being furvived by the other life. The truth of this rule is felf-evident; nor does it at all feem to depend on Mr. SIMPSON's folution in his Select Exercifes. That it is capable, however, of being deduced from the foregoing feries, may be demonftrated in the following manner. By

* I fhall juft mention the following inftances. When the age of B is 20 and that of A is 2 years, the value, by taking the joint lives from Dr. PRICE's Table, and approximating to the real values of thofe lives in the manner directed by him, Vol. II. p. 75. of his Treatife, is 29.59. When the ages of B and A are 40 and 2, 50 and 20, 60 and 2, and 70 and 10, the refpective values are 26.93, 18.78, 21.57, and 7.08, which being compared with the values in p. 344. will be found to agree almoft exactly with them.

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reasoning as in p. 335. the value of the sum S, when the ages are equal, will be $= S \times \overline{\frac{c}{br} \times \frac{b-c}{b} + \frac{\overline{b-c}|^2}{2bbr} + \frac{d}{br^2} \times \frac{c-d}{b} + \frac{\overline{c-d}|^2}{2bbr^2} + \&c.} =$ $\frac{S}{2} \times \overline{\frac{bb}{bbr} + \frac{cc}{bbr^2} + \frac{dd}{bbr^3} + \&c.} - \frac{S}{2} \times \overline{\frac{cc}{bbr} + \frac{dd}{bbr^2} + \frac{de}{bbr^3} + \&c.} = \frac{S}{2r} + \overline{\frac{S}{2r} - \frac{S}{2}}$ $\times BB$ (putting BB for the value of the two equal joint lives) $=$ $\frac{S}{2} \times \overline{\frac{1}{r} - \frac{\overline{r-1}}{r} \times BB} =$ (since the perpetuity P is $\frac{1}{r-1}$, and the interest of any sum is that sum multiplied by $r-1$) $\frac{S}{2} \times \overline{\frac{P \cdot \overline{r-1}}{r} - \frac{BB \cdot \overline{r-1}}{r}}$, which is Mr. SIMPSON's rule with Dr. PRICE's correction; that is, "half the sum multiplied by the "difference between the perpetuity and the value of the equal "joint lives, and divided by £. 1 increased by its interest for a "year."

PROBLEM III.

The ages of A and B being given; to determine the value of the sum S, payable on the extinction of one life in particular, should that happen after the extinction of the other life.

SOLUTION.

Supposing B to be the oldest of the two lives and the sum S to become payable on his decease, it is evident that this payment at the end of the first year must depend on the contingency of both lives being extinct before this period and of B's dying *last*. Retaining the same symbols, and reasoning as I have done in the solution of the first problem, this value will be expressed by the fraction $\frac{S \cdot a' \cdot \overline{b-c}}{2abr}$. The payment of the sum S at the end of the second year will depend on either of two events happening. First, that A and B both die in the second year after having survived the first, restrained, as above, to the contingency of B's having died last; secondly, that B dies in the second year and A in the first year. The value therefore of S for this year will be expressed by the two fractions $\frac{S \cdot a'' \cdot \overline{c-d}}{2abr^2} + \frac{S \cdot \overline{c-d} \cdot a'}{abr^2}$. Again, the payment of S in the third year will depend either on A and B's both dying in that year, and B having died *last*; or on B's dying in that year, and A's dying in the first or second years. The value therefore of S for this year will be $= \frac{S \cdot a''' \cdot \overline{a-e}}{2abr^3} + \frac{S \cdot \overline{d-e} \cdot \overline{a'+a''}}{abr^3}$. By proceeding in this manner for the other years the whole value of the reversion will be found $= \frac{S}{2} \times \overline{\frac{a' \cdot \overline{b-c}}{abr} + \frac{a'' \cdot \overline{c-d}}{abr^2} + \frac{a''' \cdot \overline{d-e}}{abr^3} + \frac{a'''' \cdot \overline{e-f}}{abr^4}} + \&c. + S \times \overline{\frac{a' \cdot \overline{c-d}}{abr^2} + \frac{\overline{a'+a''} \cdot \overline{d-e}}{abr^3} + \frac{\overline{a'+a''+a'''} \cdot \overline{e-f}}{abr^4} + \&c.}$ The first of these series by proceeding in the same manner as in the solution of the second problem may be found $= \frac{\beta}{2b} \times \overline{K-AK} - \frac{\overline{B-AB}}{2r} - \frac{1}{2} \cdot \overline{B-AB} + \frac{c}{2br} \times \overline{C-AC}$; and the second series may be found $= -\frac{c}{br} \times \overline{C-AC} + \frac{\overline{B-AB}}{r}$. Hence the whole value of the reversion will be $= S \times \overline{\frac{\beta r \cdot \overline{K-AK} - c \cdot \overline{C-AC}}{2br} - \frac{\overline{r-1} \cdot \overline{B-AB}}{2r}}$. Q. E. D.

Having now the value of the sum S depending on the older of the two lives dying last, the value of the same sum depending

depending on the younger of the two lives dying last is easily obtained, by subtracting the value first found from the whole value of the reversion after the extinction of both lives.

The answers computed by this rule differ rather more from those computed by Mr. SIMPSON's approximation than they do in the preceding problem. But I am fearful of becoming tedious, and therefore shall desist from inserting the comparative values in this case. It may not, however, be improper to exemplify the truth of this demonstration, by shewing in what manner the series may be resolved into the plain simple rule for computing the value of the reversion, when the ages are equal. The series in this case become $\frac{S.\overline{b-c}|^2}{2bbr}+\frac{S.\overline{c-d}|^2}{2bbr^2}$, &c. $+\frac{S.\overline{b-c}.\overline{c-d}}{bbr^2}+\frac{S.\overline{b-d}.\overline{d-e}}{bbr^3}$ + &c. = $\frac{S}{2}\times\overline{\frac{bb}{bbr}+\frac{cc-2bc}{bbr}+\frac{dd-2bd}{bbr^2}+\text{\&c.}}+\frac{S}{2}\times\overline{\frac{2bc-cc}{bbr^2}+\frac{2bd-dd}{bbr^3}}$, &c. = $S\times\overline{\frac{1}{2r}+\frac{1}{r}\times\frac{2B-BB}{2}-\frac{2B-BB}{2}}$ = (putting L for $2B-BB$, the value of an annuity on the longest of the two equal lives) $\frac{S}{2}\times\overline{\frac{1}{r}+\frac{\overline{1-r}.L}{r}}=\frac{S}{2}\times\frac{P.\overline{r-1}}{r}-\frac{L.\overline{r-1}}{r}$. Q. E. D*.

The *exact* value of all reversions depending on survivorships between two lives might be found in the same manner as the values in the preceding problems. With regard to the values of reversions depending on survivorships between three lives, I am sensible that the solutions of those cases would be rather difficult when deduced from the real probabilities of life. But they certainly might be effected; and *these* are the more necessary, inasmuch as the solutions derived from the expectations of life are often so very defective as not to deserve the name of approximations.

Chatham Place, Feb. 2, 1788.

* See the latter part of the solution of the second problem, p. 347.